

ON FINITE  $ca$ - $\mathfrak{F}$  GROUPS AND THEIR APPLICATIONS

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ABSTRACT. Let  $\mathfrak{F}$  be a class of groups. A group  $G$  is called  $ca$ - $\mathfrak{F}$ -group if its every non-abelian chief factor is simple and  $H/K \rtimes C_G(H/K) \in \mathfrak{F}$  for every abelian chief factor  $H/K$  of  $G$ . In this paper, we investigate the structure of a finite  $ca$ - $\mathfrak{F}$ -group. Properties of mutually permutable products of finite  $ca$ - $\mathfrak{F}$ -groups are studied.

## 1. INTRODUCTION

Only finite groups are considered. The concept of composition formation was introduced by L.A. Shemetkov [15] and R. Baer in an unpublished paper (noted in [7, IV, p.370]). Every saturated formation is a composition formation. The class of all quasinilpotent groups is an example of composition, but not saturated formation. Guo Wenbin and A.N. Skiba [9, 10] introduced the concept of quasi- $\mathfrak{F}$ -group that is a generalization of quasinilpotency. In [9] they proved that the class of all quasi- $\mathfrak{F}$ -groups is a composition formation if  $\mathfrak{F}$  is a saturated formation containing all nilpotent groups. In [9, 10] some applications of formations of quasi- $\mathfrak{F}$ -groups were considered.

In [19] V.A. Vedernikov introduced the definition of a  $c$ -supersoluble group. Recall [19] that a group  $G$  is called  $c$ -supersoluble if every chief factor of  $G$  is simple. In [18] A.F. Vasil'ev and T.I. Vasil'eva proved that the class  $\mathfrak{U}_c$  of all  $c$ -supersoluble groups is a composition but not a saturated formation. D. Robinson (using notation:  $SC$ -group) [13] established the structural properties of finite  $c$ -supersoluble groups.

In [12] the following generalization of  $c$ -supersolubility was proposed.

Let  $\mathfrak{F}$  be a class of groups. Recall [17] that a chief factor  $H/K$  of group  $G$  is called  $\mathfrak{F}$ -central provided  $H/K \rtimes G/C_G(H/K) \in \mathfrak{F}$ .

**Definition 1.1** ([12]). *Let  $\mathfrak{F}$  be a class of groups. A group  $G$  is called a  $ca$ - $\mathfrak{F}$ -group if its every non-abelian chief factor is simple and every abelian chief factor of  $G$  is  $\mathfrak{F}$ -central.*

The class of all  $ca$ - $\mathfrak{F}$ -groups is denoted by  $\mathfrak{F}_{ca}$ . If  $\mathfrak{F} = \mathfrak{U}$  we have that  $\mathfrak{F}_{ca} = \mathfrak{U}_c$ . If  $\mathfrak{F} = \mathfrak{NA}$ , then  $\mathfrak{F}_{ca} = (\mathfrak{NA})_{ca}$  is the class of all groups whose every non-abelian chief factor is simple and  $Aut_G(H/K)$  is abelian for every abelian chief factor  $H/K$ . If  $\mathfrak{F} = \mathfrak{S}$  then  $\mathfrak{F}_{ca}$  is the class of all SNAC-groups [13], i.e the class of all groups whose all non-abelian factors are simple.

The class of all  $ca$ - $\mathfrak{F}$ -groups is a composition formation [12]. Also in [12] some properties of the products of normal  $ca$ - $\mathfrak{F}$ -subgroups were found.

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Recall [17, §8],  $Z_{\infty}^{\mathfrak{F}}(G)$  denotes the  $\mathfrak{F}$ -hypercenter of a group  $G$ .  $Z_{\infty}^{\mathfrak{F}}(G)$  is the product of all normal subgroups  $H$  of  $G$  whose  $G$ -chief factors are  $\mathfrak{F}$ -central.

The following theorem is an extension of Robinson's result [13] for case when  $\mathfrak{F}$  is a soluble saturated formation.

**Theorem A.** *Let  $\mathfrak{F}$  be a soluble saturated formation. A group  $G$  is a  $ca$ - $\mathfrak{F}$ -group if and only if  $G$  satisfies:*

1.  $G^{\mathfrak{C}} = G^{\mathfrak{F}}$ ;
2. If  $G^{\mathfrak{C}} \neq 1$  then  $G^{\mathfrak{C}}/Z(G^{\mathfrak{C}})$  is a direct product of  $G$ -invariant non-abelian simple groups;
3.  $Z(G^{\mathfrak{C}}) \subseteq Z_{\infty}^{\mathfrak{F}}(G)$

Following Carocca [6], we say that  $G = HK$  is the mutually permutable product of subgroups  $H$  and  $K$  if  $H$  permutes with every subgroup of  $K$  and  $K$  permutes with every subgroup of  $H$ . The mutually permutable products of supersoluble and  $c$ -supersoluble subgroups were investigated in many works of different authors (see monograph [3]). A lot of papers were dedicated to the case where  $G = HK$  is the mutually permutable product of subgroups  $H$  and  $K$  which belong to a saturated formation  $\mathfrak{F}$ . Therefore we have the following problem.

**Problem.** Let  $\mathfrak{F}$  be a composition formation. What is the structure of the group  $G = HK$  where  $H$  and  $K$  are mutually permutable  $\mathfrak{F}$ -subgroups of  $G$ .

In this paper this problem is solving for a formation of  $ca$ - $\mathfrak{F}$ -groups where  $\mathfrak{F}$  is a saturated formation containing all supersoluble groups.

**Theorem B.** *Let  $\mathfrak{F}$  be a saturated formation containing the class  $\mathfrak{A}$  of supersoluble groups. Let a group  $G = HK$  be the product of the mutually permutable subgroups  $H$  and  $K$  of  $G$ . If  $G$  is a  $ca$ - $\mathfrak{F}$ -group then both  $H$  and  $K$  are  $ca$ - $\mathfrak{F}$ -groups.*

**Corollary B.1** ([4]). *Let the group  $G = HK$  be the mutually permutable product of the subgroups  $H$  and  $K$  of  $G$ . If  $G$  is a  $c$ -supersoluble group then both  $H$  and  $K$  are  $c$ -supersoluble groups.*

**Corollary B.2** ([5]). *Let the group  $G = HK$  be the mutually permutable product of the subgroups  $H$  and  $K$  of  $G$ . If  $G$  is a SNAC-group then both  $H$  and  $K$  are SNAC-groups.*

**Corollary B.3.** *Let the group  $G = HK$  be the mutually permutable product of the subgroups  $H$  and  $K$  of  $G$ . If  $G$  is a  $ca$ - $\mathfrak{NA}$ -group then both  $H$  and  $K$  are  $ca$ - $\mathfrak{NA}$ -groups.*

It is well known that in general, the product  $G = HK$  of two normal supersoluble subgroups of a finite group  $G$  need not be supersoluble. In 1957 Baer [2] established that such group  $G$  will be a supersoluble if and only if the derived subgroup  $G'$  of  $G$  is nilpotent. The next theorem is an extension of this result.

**Theorem C.** *Let  $\mathfrak{F}$  be a saturated formation containing the class  $\mathfrak{A}$  of supersoluble groups. Let the group  $G = HK$  be the product of the mutually permutable  $ca$ - $\mathfrak{F}$ -subgroups  $H$  and  $K$  of  $G$ . If the derived subgroup  $G'$  of  $G$  is quasinilpotent, then  $G$  is a  $ca$ - $\mathfrak{F}$ -group.*

**Corollary C.1** ([4]). *Let the group  $G = HK$  be the product of the mutually permutable  $c$ -supersoluble subgroups  $H$  and  $K$  of  $G$ . If the derived subgroup  $G'$  of  $G$  is quasinilpotent, then  $G$  is  $c$ -supersoluble.*

**Corollary C.2** ([1]). *Let the group  $G = HK$  be the product of the mutually permutable supersoluble subgroups  $H$  and  $K$  of  $G$ . If the derived subgroup  $G'$  of  $G$  is nilpotent, then  $G$  is supersoluble.*

**Corollary C.3.** *Let the group  $G = HK$  be the product of the mutually permutable  $ca\text{-}\mathfrak{NA}$ -subgroups  $H$  and  $K$  of  $G$ . If the derived subgroup  $G'$  of  $G$  is quasinilpotent, then  $G$  is  $ca\text{-}\mathfrak{NA}$ -group.*

The following corollary extends [12] the properties of normal products of  $ca\text{-}\mathfrak{F}$ -groups.

**Corollary C.4.** *Let  $\mathfrak{F}$  be a saturated formation containing the class  $\mathfrak{U}$  of supersoluble groups. If  $G = HK$  is the product of normal  $ca\text{-}\mathfrak{F}$ -subgroups  $H$  and  $K$  of  $G$  and the derived subgroup  $G'$  of  $G$  is quasinilpotent, then  $G$  is a  $ca\text{-}\mathfrak{F}$ -group.*

## 2. PRELIMINARIES

Standard notations, notions and results are used in the paper (see [7, 16]). Recall significant notions and notations for this paper.  $\mathbb{P}$  is the set of all prime numbers;  $1$  is an identity group;  $H \rtimes K$  is a semidirect product of groups  $H$  and  $K$ ;  $\mathfrak{G}$  is the class of all groups;  $\mathfrak{S}$  is the class of all soluble groups;  $\mathfrak{U}$  is the class of all supersoluble groups;  $\mathfrak{U}_c$  is the class of all  $c$ -supersoluble groups;  $\mathfrak{N}$  is the class of nilpotent groups;  $\mathfrak{N}_p$  is the class of all  $p$ -groups;  $\mathfrak{J}$  is the class of all simple groups;  $\mathfrak{A}(p-1)$  is the class of all abelian groups of exponent dividing  $p-1$ .

A *formation* is a homomorph  $\mathfrak{F}$  of groups such that each group  $G$  has the smallest normal subgroup (called  $\mathfrak{F}$ -residual and denoted by  $G^{\mathfrak{F}}$ ) with quotient in  $\mathfrak{F}$ . A formation  $\mathfrak{F}$  is said to be *saturated* if it contains each group  $G$  with  $G/\Phi(G) \in \mathfrak{F}$ . A formation  $\mathfrak{F}$  is said to be (normally) *hereditary* if it contains all (normal) subgroups of every group in  $\mathfrak{F}$ .

Let  $\mathfrak{F}$  be a non-empty formation.  $G_{\mathfrak{F}}$  denotes  $\mathfrak{F}$ -radical of group  $G$ , i.e., the largest normal  $\mathfrak{F}$ -subgroup of  $G$ .

A function  $f : \mathbb{P} \rightarrow \{\text{formations of groups}\}$  is called a *local formation function*. The symbol  $LF(f)$  denotes the class of all groups such that either  $G = 1$  or  $G \neq 1$  and  $G/C_G(H/K) \in f(p)$  for every chief factor  $H/K$  of  $G$  and every  $p \in \pi(H/K)$ . The class  $LF(f)$  is a non-empty formation.

For a formation  $\mathfrak{F}$ , if there exists a formation function  $f$  such that  $\mathfrak{F} = LF(f)$  then  $\mathfrak{F}$  is called a *local formation*. It is known that a formation  $\mathfrak{F}$  is local if and only if it is saturated [7, IV, Theorem 4.6].

A formation  $\mathfrak{F}$  is said to be *solubly saturated*, *composition*, or *Baer-local formation* if it contains each group  $G$  with  $G/\Phi(N) \in \mathfrak{F}$  for some soluble normal subgroup  $N$  of  $G$ . For every function  $f$  of the form  $f : \mathfrak{J} \rightarrow \{\text{formations of groups}\}$  we put,  $CLF(f) = \{G \text{ is a group} \mid G/C_G(H/K) \in f(A) \text{ for every } A \in \mathcal{K}_{H/K}\}$ . It is well known that a *composition formation* (or a Baer-local formation if we use the terminology in [7])  $\mathfrak{F}$  is exactly a class  $\mathfrak{F} = CLF(f)$  for some function  $f$  of above-mention form. In this case, the function  $f$  is said to be a *composition satellite* [14] of the formation  $\mathfrak{F}$ .

A local function  $f$  is called an *inner local function* if  $f(p) \subseteq LF(f)$  for every prime  $p$ . Function  $f$  is called a *maximal inner local function* of formation  $\mathfrak{F}$  if  $f$  is a maximal element of set of all inner local functions of formation  $\mathfrak{F}$ . Similarly, we can introduce the notion of the inner composition satellite and maximal inner composition satellite.

Every local (composition) formation has the unique maximal inner local function (composition satellite) [16, ch. 1].

We will use the following results.

**Lemma 2.1** ([12]). *Let  $\mathfrak{F}$  be a class of groups. Then the class  $\mathfrak{F}_{ca}$  is a non-empty formation.*

**Theorem 2.2** ([12]). *Let  $\mathfrak{F}$  be a saturated formation and  $f$  is its maximal inner local function. Then the formation  $\mathfrak{F}_{ca}$  is a composition formation and has a maximal inner composition satellite  $h$  such that  $h(N) = \mathfrak{F}_{ca}$ , if  $N$  is a non-abelian group and  $h(N) = f(p)$ , if  $N$  is a simple  $p$ -group, where  $p$  is a prime.*

**Lemma 2.3** ([18]). *Let  $\mathfrak{F}$  be a formation and  $N$  be a minimal normal subgroup of  $G$  such as  $|N| = p^a$  for some prime  $p$ . If  $N$  contains in the subgroup  $H$  of  $G$  and  $H/C_H(U/V) \in \mathfrak{F}$  for every  $H$ -chief factor  $U/V$  of  $N$ , then  $H/C_H(N) \in \mathfrak{N}_p \mathfrak{F}$ .*

**Lemma 2.4** ([3]). *Assume that the subgroups  $A$  and  $B$  of the group  $G$  are mutually permutable and that  $N$  is a normal subgroup of  $G$ . Then the subgroups  $AN/N$  and  $BN/N$  are mutually permutable in  $G/N$ .*

**Lemma 2.5** ([3]). *Let the group  $G = AB$  be the mutually permutable product of the subgroups  $A$  and  $B$ . Then:*

1. *If  $N$  is a maximal normal subgroup of  $G$ , then  $\{AN, BN, (A \cap B)N\} \subseteq \{N, G\}$ .*
2. *If  $N$  is a non-abelian minimal normal subgroup of  $G$ , then  $\{A \cap N, B \cap N\} \subseteq \{N, 1\}$  and  $N = (N \cap A)(N \cap B)$  (that is,  $N$  is prefactorised with respect to  $G = AB$ ).*
3. *If  $N$  is a minimal normal subgroup of  $G$ , then  $N \leq A \cap B$  or  $[N, A \cap B] = 1$ .*
4. *If  $N$  is a minimal normal subgroup of  $G$ , then  $\{A \cap N, B \cap N\} \subseteq \{N, 1\}$ .*
5. *If  $N$  is a minimal normal subgroup of  $G$  contained in  $A$  and  $B \cap N = 1$ , then  $N \leq C_G(A)$  or  $N \leq C_G(B)$ . If furthermore  $N$  is not cyclic, then  $N \leq C_G(B)$ .*

**Lemma 2.6** ([3]). *Let the group  $G = AB$  be the product of the mutually permutable subgroups  $A$  and  $B$  and let  $\mathfrak{F}$  be a saturated formation containing the class  $\mathfrak{U}$  of all supersoluble groups. If  $(A \cap B)_G = 1$ , then  $G \in \mathfrak{F}$  if and only if  $A \in \mathfrak{F}$  and  $B \in \mathfrak{F}$ .*

### 3. PROOF OF THEOREM A

In this section we prove the theorem that describes the structure of finite  $ca$ - $\mathfrak{F}$ -group.

**Lemma 3.1.** . *Let  $\mathfrak{F}$  be a soluble formation containing the class  $\mathfrak{U}$  of all supersoluble groups. If  $G$  is a  $ca$ - $\mathfrak{F}$ -group then the following statements hold:*

1.  $G^\mathfrak{F} \leq C_G(G_\mathfrak{F})$ ;
2.  $(G^\mathfrak{F})_\mathfrak{F} \leq Z(G^\mathfrak{F})$ .

*Proof.* Prove the statement 1. Obviously that all chief factors of group  $G$  below subgroup  $G_\mathfrak{F}$  are  $\mathfrak{F}$ -central. Hence subgroup  $G_\mathfrak{F}$  is  $\mathfrak{F}$ -hypercentral and thus it is subgroup of  $\mathfrak{F}$ -hypercentre  $Z_\infty^\mathfrak{F}(G)$ . By Corollary 9.3.2 [16] we have that  $G^\mathfrak{F} \leq C_G(Z_\infty^\mathfrak{F}(G))$ . Since  $\mathfrak{F}$  is a soluble formation,  $G^\mathfrak{F} \leq G^\mathfrak{F} \leq C_G(G_\mathfrak{F})$ .

Prove the statement 2. Let  $R = (G^\mathfrak{F})_\mathfrak{F}$ . Since  $R \text{ char } G^\mathfrak{F} \trianglelefteq G$ , it follows  $R \trianglelefteq G$ . Therefore  $R \leq G_\mathfrak{F}$ . Hence  $G^\mathfrak{F} \leq C_G(R)$  by statement 1 of the Lemma. The statement 2 is true.  $\square$

*Proof of theorem A.* Denote by  $D$  the soluble residual  $G^\mathfrak{F}$  of group  $G$ .

Let  $G$  be a  $ca$ - $\mathfrak{F}$ -group. If  $G$  is soluble, then  $D = 1$  and  $G \in \mathfrak{F}$ . So  $G$  satisfies the Statements 1, 2, and 3. We assume that group  $G$  is not soluble. Then  $D \neq 1$ .

Since  $\mathfrak{F}$  is a soluble formation,  $G/G^\mathfrak{F} \in \mathfrak{F} \subseteq \mathfrak{S}$ . Hence  $D \subseteq G^\mathfrak{F}$ . Since  $\mathfrak{F}_{ca}$  is a formation, it follows  $G/D \in \mathfrak{F}_{ca}$ . By solvability of quotient  $G/D$  we have that  $G/D \in \mathfrak{F}$ . Hence  $G^\mathfrak{F} \subseteq D$  and  $D = G^\mathfrak{F}$ . The Statement 1 holds. Note that all chief factors of  $G$  below  $Z(D)$  are abelian and therefore are  $\mathfrak{F}$ -central. This means, that  $Z(D)$  is  $\mathfrak{F}$ -hypercentral and the Statement 3 holds.

We show that  $D/Z(D)$  is a direct product of  $G$ -invariant simple groups.

Assume that  $Z(D) = 1$ . Let  $N_1$  be a minimal normal subgroup of  $G$  contained in  $D$ . If  $N_1$  is abelian, then it follows from  $N_1 \leq D_\mathfrak{S}$  and the Statement 2 of Lemma 3.1 that  $N_1 \leq Z(D) = 1$ . Hence  $N_1$  is non-abelian. Since  $G \in \mathfrak{F}_{ca}$ , we have that  $N_1$  is a simple. Note that  $G/C_G(N_1)$  is isomorphic to a subgroup of  $\text{Aut}(N_1)$  and  $N_1 C_G(N_1)/C_G(N_1)$  is isomorphic to  $\text{Inn}(N_1)$ . So  $G/N_1 C_G(N_1) \simeq (G/C_G(N_1))/(N_1 C_G(N_1)/C_G(N_1))$  is isomorphic to a subgroup of  $\text{Aut}(N_1)/\text{Inn}(N_1)$ . From the validity of the Schreier conjecture, it follows that  $G/N_1 C_G(N_1)$  is soluble. Then  $D \leq N_1 C_G(N_1)$ . Hence  $D = D \cap N_1 C_G(N_1) = N_1(D \cap C_G(N_1)) = N_1 C_D(N_1)$  and  $N_1 \cap C_D(N_1) = 1$ . If  $D = N_1$ , then the Statement 2 holds. Assume that  $D$  is not simple. Therefore,  $C_D(N_1) \neq 1$ . The Statement 2 holds in the case when  $C_D(N_1)$  is simple. Assume that  $C_D(N_1)$  is not a simple and let  $N_2$  be a minimal normal subgroup of  $G$  contained in  $C_D(N_1)$ . Since  $Z(D) = 1$  and the Statement 2 of lemma 3.1, it follows that  $N_2$  is a simple non-abelian subgroup. By the above  $D = N_2 C_D(N_2)$ . By Dedekind identity  $C_D(N_1) = C_D(N_1) \cap N_2 C_D(N_2) = N_2(C_D(N_1) \cap C_D(N_2)) = N_2 C_L(N_2)$ , where  $L = C_D(N_1) \cap C_D(N_2)$ . Then  $D = N_1 N_2 C_L(N_2)$ . Applying above to  $C_D(N_2)$  and etc. we can conclude that  $D = N_1 \times N_2 \times \cdots \times N_t$  is the direct product of minimal normal subgroups of  $G$ , each of them simple, as desired. So the Statement 2 holds.

Let  $Z(D) \neq 1$ . Since  $G/Z(D) \in \mathfrak{F}_{ca}$  and  $(G/Z(D))^\mathfrak{S} = D/Z(D)$ , the Statement 1 and 3 holds for  $G/Z(D)$ . Denote  $T/Z(D) = Z(D/Z(D))$ . Then  $T$  is a normal soluble subgroup of  $D$ . By lemma 3.1  $T$  is contained in the center  $Z(D)$ . Therefore  $Z(D/Z(D)) = 1$ . By the above the Statement 1 holds for  $G/Z(D)$ .

Conversely, assume that a group  $G$  satisfies the Statements 1, 2 and 3. We consider a chief series of  $G$  which passes through the subgroup  $D = G^\mathfrak{S}$ . Note the all chief factors above  $D$  are abelian and  $\mathfrak{F}$ -central. By the Statement 2 the quotient  $D/Z(D)$  is the direct product of minimal normal subgroups of  $G/Z(D)$ , which are simple. All chief factors of  $G$  below  $Z(D)$  are  $\mathfrak{F}$ -central by the Statement 3. By virtue of Jordan-Holder's theorem for groups with operators [7, A, 3.2] and the Definition 1.1,  $G \in \mathfrak{F}_{ca}$ .  $\square$

#### 4. PROOF OF THEOREM B AND C

In this section we prove some properties of the mutually permutable products of  $ca$ - $\mathfrak{F}$ -groups.

*Proof of Theorem B.* Assume that that this theorem is false and let  $G$  be a counterexample of minimal order. Let  $N$  be a minimal normal subgroup of  $G$ . If  $N = G$ , then  $G$  is simple. Hence  $H \in \mathfrak{F}_{ca}$  and  $K \in \mathfrak{F}_{ca}$ . Assume  $N \neq G$ . By Lemma 2.4  $G/N = HN/N \cdot KN/N$  is the mutually permutable product of subgroups  $HN/N$  and  $KN/N$  of  $G/N$ . Note that  $G/N \in \mathfrak{F}_{ca}$ . Then all conditions of the theorem hold for  $G/N$ . Therefore  $HN/N \simeq H/(H \cap N) \in \mathfrak{F}_{ca}$  and  $KN/N \simeq K/(K \cap N) \in \mathfrak{F}_{ca}$ .

Since  $\mathfrak{F}_{ca}$  is a formation by Lemma 2.1, it follows that  $N$  is the unique minimal normal subgroup of  $G$ .

Let  $N$  be a non-abelian group. Then  $N$  is simple. According to Lemma 2.5 we should consider the following cases.

1. Let  $H \cap N = K \cap N = N$ . Then  $N \leq H \cap K$ ,  $H/(H \cap N) = H/N \in \mathfrak{F}_{ca}$  and  $K/(K \cap N) = K/N \in \mathfrak{F}_{ca}$ . Hence  $H$  and  $K$  are  $ca$ - $\mathfrak{F}$ -groups, a contradiction.
2. Let  $H \cap N = K \cap N = 1$ . Then  $H/(H \cap N) \simeq H$  and  $K/(K \cap N) \simeq K$  are  $ca$ - $\mathfrak{F}$ -groups, a contradiction.
3. Let  $H \cap N = N$  and  $K \cap N = 1$ . Then  $H/(H \cap N) = H/N \in \mathfrak{F}_{ca}$  and  $H$  is a  $ca$ - $\mathfrak{F}$ -group and  $K/(K \cap N) \simeq K$  is a  $ca$ - $\mathfrak{F}$ -group. A contradiction.
4. Let  $H \cap N = 1$  and  $K \cap N = N$ . This case is considered similarly to the case 3.  $\square$

To prove the Theorem C we need the following results.

**Lemma 4.1.** *Let the group  $G$  has the unique minimal normal subgroup  $N = N_1 \times \dots \times N_t$  and  $N_i$  are isomorphic simple non-abelian groups for all  $i = 1, \dots, t$ . If  $N \subseteq H$ , where  $H$  is a  $ca$ - $\mathfrak{F}$ -subgroup of  $G$ , then  $N_i \triangleleft H$  for all  $i = 1, \dots, t$ .*

*Proof.* Let  $i \in \{1, \dots, t\}$ . Consider normal closure  $N_i^H = \langle N_i^x | x \in H \rangle$  of subgroup  $N_i$  in  $H$ . Note that  $N_i \triangleleft \triangleleft G$ . Hence  $N_i \triangleleft \triangleleft H$ . By the Lemma 9.17 [11] we have that  $N_i^H$  is a minimal normal subgroup of  $H$ . Since subgroup  $N_i^H$  is non-abelian and isomorphic to the chief factor of  $ca$ - $\mathfrak{F}$ -subgroup  $H$ , then  $N_i^H$  is simple. Then, by  $N_i \triangleleft \triangleleft N_i^H$  we have that  $N_i^H = N_i$ . Hence  $N_i \triangleleft H$  for all  $i = 1, \dots, t$ .  $\square$

**Lemma 4.2.** *Let  $\mathfrak{F}$  be a composition formation and  $f$  is an inner composition satellite of  $\mathfrak{F}$ . Let a group  $G$  has the unique minimal normal subgroup  $N$  and  $N$  is an abelian  $p$ -group for some prime  $p$ . The chief factor  $N$  of  $G$  is  $\mathfrak{F}$ -central in  $G$  if and only if  $G/C_G(N) \in f(p)$ .*

*Proof.* Let  $G/C_G(N) \in f(p)$ . Consider semidirect product  $R = N \rtimes G/C_G(N)$ . Note that  $N$  is the unique minimal normal subgroup of  $R$  and  $C_R(N) = N$ . Then  $R/C_R(N) \simeq G/C_G(N) \in f(p) \subseteq \mathfrak{F}$ . Hence  $R \in \mathfrak{F}$ , i.e. the chief factor  $N/1$  of  $G$  is  $\mathfrak{F}$ -central.

Conversely, assume that  $N$  is  $\mathfrak{F}$ -central chief factor of  $G$ . Then  $R = N \rtimes G/C_G(N) \in \mathfrak{F}$ , where  $N$  is the unique minimal normal subgroup of  $R$  and  $C_G(N) = N$ . Hence  $R/C_R(N) \simeq G/C_G(N) \in f(p)$ .  $\square$

**Lemma 4.3.** *Let  $\mathfrak{F}$  be a formation and  $\mathfrak{A}(p-1) \subseteq \mathfrak{F}$ . Let  $G = HK$  be the mutually permutable products of subgroup  $H$  and  $K$ , where  $H, K \in \mathfrak{N}_p\mathfrak{F}$  and  $G \in \mathfrak{N}_p\mathfrak{A}$ . Then  $G \in \mathfrak{N}_p\mathfrak{F}$ .*

*Proof.* Assume that this lemma is false and let  $G$  be a counterexample of minimal order. Let  $N$  be a minimal normal subgroup of  $G$ . We can assume without loss of generality that  $G \neq N$ . By Lemma 2.4  $G/N = HN/N \cdot KN/N$  is the mutually permutable product of subgroups  $HN/N$  and  $KN/N$  of  $G/N$ . Note that  $HN/N \in \mathfrak{N}_p\mathfrak{F}$ ,  $HN/N \in \mathfrak{N}_p\mathfrak{F}$  and  $G/N \in \mathfrak{N}_p\mathfrak{A}$ . Then all conditions of

the Lemma 4.3 hold for  $G/N$ . Therefore  $G/N \in \mathfrak{N}_p\mathfrak{F}$ . Since  $\mathfrak{N}_p\mathfrak{F}$  is a formation, it follows that  $N$  is the unique minimal normal subgroup of  $G$ . We note that  $N$  is a  $q$ -group for some prime  $q \neq p$ . Since  $G \in \mathfrak{N}_p\mathfrak{A}$  and  $O_p(G) = 1$ , we have that  $G \in \mathfrak{A}$ . Therefore  $H \in \mathfrak{F}$  and  $K \in \mathfrak{F}$ . Since  $N$  is the unique minimal normal subgroup of  $G$ , it follows that  $G$  is a cyclic  $q$ -group. Since  $G = HK$ , we have that  $G = H$  or  $G = K$ , i.e.  $G \in \mathfrak{F}$ .  $\square$

*Proof of Theorem C.* Assume that this theorem is false and let  $G$  be a counterexample of minimal order. Let  $N$  be a minimal normal subgroup of  $G$ . If  $N = G$ , then  $G$  is simple. Hence  $G \in \mathfrak{F}_{ca}$ . Assume  $N \neq G$ . By Lemma 2.4  $G/N = HN/N \cdot KN/N$  is the mutually permutable product of subgroups  $HN/N$  and  $KN/N$  of  $G/N$ . Note that the derived subgroup  $(G/N)'$  of  $G/N$  is quasinilpotent. Then all conditions of the Theorem hold for  $G/N$ . Therefore  $G/N \in \mathfrak{F}_{ca}$ . Since  $\mathfrak{F}_{ca}$  is a formation by Lemma 2.1, it follows that  $N$  is the unique minimal normal subgroup of  $G$ .

Let  $N$  be a non-abelian group. Then  $N = N_1 \times \cdots \times N_t$ , where  $N_i$  are isomorphic simple non-abelian groups for all  $i = 1, \dots, t$ . According to Lemma 2.5 we should consider the following cases.

1. Let  $H \cap N = K \cap N = N$ . Then  $N \subseteq H \cap K$ . Since  $H$  and  $K$  are  $ca$ - $\mathfrak{F}$ -subgroups and  $N = N_1 \times \cdots \times N_t$ , it follows that  $N_i \triangleleft H$  and  $N_i \triangleleft K$  by Lemma 4.1. Hence by  $G = HK$  we have that  $N_i \triangleleft G$  for all  $i = 1, \dots, t$ . Since  $N$  is the unique minimal normal subgroup of  $G$ , it follows that  $t = 1$  and  $N$  is simple. Since  $G/N \in \mathfrak{F}_{ca}$ , we have that  $G \in \mathfrak{F}_{ca}$ . A contradiction.

2. Let  $H \cap N = K \cap N = 1$ . Then  $N = (H \cap N)(K \cap N) = 1$  by Lemma 2.5(2). A contradiction with choice of  $N$ .

3. Let  $H \cap N = N$  and  $K \cap N = 1$ . Then  $N \subseteq H$  and  $N \leq C_G(H)$  or  $N \leq C_G(K)$  by Lemma 2.5(5). Since  $N \leq H$  and  $N$  is non-abelian, we have that  $N \leq C_G(K)$ . Since  $N = N_1 \times \cdots \times N_t$ , it follows that  $N_i \leq C_G(K)$  for all  $i = 1, \dots, t$ . By  $N_i \leq H$  we have that  $N_i \triangleleft H$  for all  $i = 1, \dots, t$  by Lemma 4.1. By  $G = HK$  we have that  $N_i \triangleleft G$ . Since  $N$  is the unique minimal normal subgroup of  $G$ , it follows that  $N = N_i$  and  $N$  is simple. Since  $G/N \in \mathfrak{F}_{ca}$ , we have that  $G \in \mathfrak{F}_{ca}$ . A contradiction.

4. Let  $H \cap N = 1$  and  $K \cap N = N$ . This case is considered similarly to the case 3.

Assume  $N$  is an abelian group. Then  $N$  is a  $p$ -group for some prime  $p$ . By Theorem 2.2 formation  $\mathfrak{F}_{ca}$  has the maximal inner composition satellite  $h$  such that  $h(N) = f(p)$ , where  $f$  is a maximal inner local function of  $\mathfrak{F}$ . According to Lemma 2.5 we should consider the following cases.

1. Let  $H \cap N = K \cap N = N$ . Then  $N \subseteq H \cap K$ . Let  $U/V$  is any  $H$ -chief factor of  $N$ . Since  $H \in \mathfrak{F}_{ca}$ , it follows that  $H/C_H(U/V) \in h(p)$ . By Lemma 2.3 we have that  $H/C_H(N) \in \mathfrak{N}_p h(p) = h(p)$ . Similarly we can show that  $K/C_K(N) \in \mathfrak{N}_p h(p) = h(p)$ . Note the group  $G/C_G(N) = HC_G(N)/C_G(N) \cdot KC_G(N)/C_G(N)$  is the mutually permutable product of subgroups  $HC_G(N)/C_G(N)$  and  $KC_G(N)/C_G(N)$  of  $G/C_G(N)$ .

Since  $N \subseteq G'$  and  $G'$  is quasinilpotent, it follows that  $G'/C_{G'}(N) \in \mathfrak{N}_p$  by Lemma 2.3. So  $(G/C_G(N))' = G'C_G(N)/C_G(N) \simeq G'/C_{G'}(N)$  is a  $p$ -group. Since  $G/C_G(N)/(G/C_G(N))' \in \mathfrak{A}$ , it follows that  $G/C_G(N) \in \mathfrak{N}_p \mathfrak{A}$ . By Lemma 4.3 for  $G/C_G(N)$  we have that  $G/C_G(N) \in \mathfrak{N}_p h(p) = h(p) \subseteq \mathfrak{F}_{ca}$ . Therefore  $G \in \mathfrak{F}_{ca}$ . A contradiction.

2. Let  $H \cap N = K \cap N = 1$ . Then  $N \not\subseteq H \cap K$  and  $(H \cap K)_G = 1$ . If  $H^\mathfrak{S} = 1$  and  $K^\mathfrak{S} = 1$ , then  $H$  and  $K$  are soluble. Hence  $H \in \mathfrak{F}$  and  $K \in \mathfrak{F}$ . By Lemma 2.6 we have that  $G \in \mathfrak{F} \subseteq \mathfrak{F}_{ca}$ , a contradiction. Hence we can assume without loss of generality that  $H^\mathfrak{S} \neq 1$ . Then  $H^\mathfrak{S} \triangleleft G$  by Corollary 4.3.6 [3]. Hence  $N \leq H^\mathfrak{S}$ . Therefore  $N \leq H \cap N = 1$ , a contradiction.

3. Let  $H \cap N = N$  and  $K \cap N = 1$ . Assume that  $N$  is non-cyclic subgroup. Then  $N \leq C_G(K)$  by Lemma 2.5(5). Hence  $K \subseteq C_G(N)$  and  $G/C_G(N) =$

$HC_G(N)/C_G(N) \cdot KC_G(N)/C_G(N) = HC_G(N)/C_G(N) \simeq H/(C_G(N) \cap H) = H/C_H(N)$ . Since  $N \subseteq H$  and  $H \in \mathfrak{F}_{ca}$ , it follows that  $H/C_H(N) \in h(p)$  by Lemma 2.3. By Lemma 4.2 we have that factor  $N$  is  $\mathfrak{F}$ -central chief factor of  $G$ . Then  $G \in \mathfrak{F}_{ca}$ , a contradiction. Let  $N$  be a cyclic group. Then  $|N| = p$  and  $G/C_G(N)$  is a cyclic group of order dividing  $p-1$ . Hence  $G/C_G(N) \in \mathfrak{A}(p-1) \subseteq f(p) = h(p)$ . Since  $G/N \in \mathfrak{F}_{ca}$ , it follows that  $G \in \mathfrak{F}_{ca}$ , a contradiction.

4. Let  $H \cap N = 1$  and  $K \cap N = N$ . This case is considered similarly to the case 3.  $\square$

## 5. FINAL REMARKS

Many different specific examples of composition formations containing all supersoluble groups can be built using the concept of  $ca$ - $\mathfrak{F}$ -group.

According to [7], a *rank function* is a map  $R : \mathbb{P} \rightarrow \mathcal{R}(p)$  which associates with each prime  $p$  a set  $R(p)$  of natural numbers. With each rank function  $R$  we associate a class [7]

$$\mathfrak{F}(R) = (G \in \mathfrak{S} \mid \text{for all prime } p \in \mathbb{P} \text{ each } p\text{-factor of } G \text{ has rank in } R(p)),$$

that is a formation.

If  $\mathfrak{F}(R)$  is a saturated formation, then rank function is called a *saturated* (see [7, p. 484]). A rank function  $R$  is said to have *full characteristic* if  $R(p) \neq \emptyset$  for all  $p \in \mathbb{P}$ .

Note that if  $R$  is a saturated rank function of full characteristic, by [7, IV, 4.3], we have  $1 \in R(p)$  for all prime  $p \in \mathbb{P}$  and therefore  $\mathfrak{U} \subseteq \mathfrak{F}(R)$ .

If a rank function  $R$  is defined, then for all prime  $p \in \mathbb{P}$  are defined [7]

$$\begin{aligned} \pi(G) &= R(p) \cap \mathbb{P} \text{ and} \\ e(p) &= \{p^m - 1 \mid m \in R(p)\}. \end{aligned}$$

By  $\mathfrak{A}_{\pi(p)'}(e(p))$  we denote a class of abelian  $\pi(p)'$ -group with exponent dividing  $e(p)$  that is a formation.

According to [7] the following lemma holds.

**Lemma 5.1** ([7]). *Let  $R$  is a saturated rank function of full characteristic. Then  $R$  satisfies the following conditions*

**RF1:** *If  $n \in R(p)$  and  $m \mid n$ , then  $m \in R(p)$ ;*

**RF2:** *If  $\{m, n\} \in R(p)$ , then  $mn \in R(p)$ ;*

**RF3:** *If  $p$  and  $q$  are distinct primes with  $q \in R(p)$  and if  $m \in R(p)$ , then  $q^m - 1 \in R(p)$ ;*

**RF4:** *If  $p, q \in \mathbb{P}$  and  $r \in \mathbb{N}$  satisfy the following conditions:*

- (i)  $p \mid (q^m - 1)$  for some  $m \in R(p)$ ,
  - (ii)  $q \mid (p^n - 1)$  for some  $n \in R(p)$ ,
  - (iii)  $r \mid (p^k - 1)$  for some  $k \in R(p)$ ,
  - (iv)  $p \in R(p)$ ,  $r \in R(p)$ ,
- then  $r \in R(p)$ .*

Local function of formation  $\mathfrak{F}(R)$  in the case when  $R$  is a saturated rank function is described in theorem 2.18 [7, p. 490] which we form as lemma.

**Lemma 5.2.** *Let  $R$  is a rank function and let  $\hat{\mathfrak{F}}(R)$  is a local formation defined by local function  $f$  such that  $f(p) = \mathfrak{A}_{\pi(p)'}(e(p))\mathfrak{S}_{\pi}(p)$  for all prime  $p$ . Then any two of the following statements are equivalent:*

- (a)  *$R$  is a saturated rank function;*



- (b)  $R$  satisfies Conditions **RF1-RF4**;
- (c)  $\hat{\mathfrak{F}} = \mathfrak{F}$ .

**Corollary B.4.** *Let  $R$  be a saturated rank function of full characteristic and the group  $G = HK$  be the mutually permutable product of the subgroups  $H$  and  $K$  of  $G$ . If  $G$  is a  $ca\text{-}\mathfrak{F}(R)$ -group, then  $H$  and  $K$  are also  $ca\text{-}\mathfrak{F}(R)$ -groups.*

**Corollary C.5.** *Let  $R$  be a saturated rank function of full characteristic and the group  $G = HK$  be the mutually permutable product of the  $ca\text{-}\mathfrak{F}(R)$ -subgroups  $H$  and  $K$  of  $G$ . If the derived subgroup  $G'$  of  $G$  is quasinilpotent, then  $G$  is a  $ca\text{-}\mathfrak{F}(R)$ -group.*

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